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1984 J. Phys. A: Math. Gen. 17 3315

(http://iopscience.iop.org/0305-4470/17/17/008)

J. Phys. A: Math. Gen. 17 (1984) 3315-3320. Printed in Great Britain

Solitons and wavetrains: unified approach

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Received 8 December 1983, in final form 20 March 1984

Abstract. The direct employment of the T-function defined as a limiting transformation of the Riemann theta function leading to a system of algebraic dispersion equations, yields solutions of some nonlinear partial differential equations in the form of solitons on a background of quasi-periodic waves. Pure solitons and pure quasi-periodic solutions appear as particular cases. The method is illustrated by the example of the KdV equation and is also compared with the Hirota bilinear operator technique.

1. Introduction and definitions

In this paper we present a unified approach to the soliton and quasi-periodic solutions of some nonlinear partial differential equations (NLPDE) which, we assume, have quasi-periodic solutions which can be expressed by the Riemann abstract theta function (θ-f). In this method solutions in the form of solitons on a background of quasi-periodic waves can automatically appear.

The problem is solved by an application of what we call the T-function (T-f). This in some sense generalises the θ-f and has already been applied in a similar form to the sine-Gordon equation (Zagrodziński and Jaworski 1982).

One can expect that the quasi-periodic solution would tend to the soliton solution when all the diagonal elements of the B-matrix (which is a parameter) increase. This idea was expressed by Matveev (1979) who discussed the spectra of the associated scattering problems, but to our knowledge it has never been exploited. Such a limit, if it exists, we shall call the 'complete soliton limit'. Then also the 'partial soliton limits' would exist when only some diagonal elements of the B-matrix tend to infinity.

Quite analogously, the T-f as a complete and/or partial limit of the Riemann θ-f is defined. Our approach is based on a different principle from that of the generalisation presented in the series of papers by Jimbo *et al* (1981), although the resemblances between their τ-function and our T-f are quite marked.

Another difference is that we employ a system of dispersion equations for the NLPDE being considered which allows us to determine all the parameters of the solutions effectively (Dubrovin 1981, Zagrodzinski 1982). Although the method presented here can be applied to all KdV-type, KP, sG, Toda and related equations, we confine ourselves here to the simple KdV equation (for some other equations see appendix 2).

For the fixed decomposition of the g-dimensional complex space $C^g = C^s \times C^p$ we define the T-f, as in Zagrodziński and Jaworski (1982) by the limiting procedure

$$T\begin{pmatrix} z^s \\ z^p \end{pmatrix} := \lim_{D^{ss}\to\infty} \theta\left(\begin{matrix} z^s - \mathrm{i}D^{ss}e^s/2 \\ z^p \end{matrix}\middle| B\right), \tag{1}$$

or

$$T(z) := \sum_{\alpha \in D^s} \exp[i\pi(2\langle\alpha, z^s\rangle + \langle\alpha, \bar{B}^{ss}\alpha\rangle)]\theta(z^p + B^{ps}\alpha \mid B^{pp}), \tag{2}$$

where $z \in C^g$, $z^{s,p} \in C^{s,p}$, respectively; $\theta(z)$ is the abstract Riemann theta function of the argument $z = z^s + z^p$, (Igusa 1972, Dubrovin 1981); $B \in C^{g\times g}$ is the Riemannian matrix to be thought of here as a parameter and is composed of blocks B^{ss}, B^{sp}, B^{ps}, B^{pp}. The quantities D^{ss} are defined as $D^{ss} = \mathrm{Diag}(\mathrm{Im}\, B^{ss})$, then $\bar{B}^{ss} = B^{ss} - iD^{ss}$, $D^s \subset Z^s$, $D^g \subset Z^g$ represent the unit hypercubes in the relevant lattices and e^s is the unit vector in D^s (see appendix 1 for details).

For the T-f the following addition theorem holds

$$T(z-w)T(z+w) = 2\sum_{\alpha \in D^g} \Omega_\alpha(w) T^2(z+\alpha/2), \tag{3}$$

where

$$\Omega_\alpha(w) = 2^{-(g+1)} \sum_{\delta \in D^g} (-1)^{\langle\alpha,\delta\rangle}[R(0|\delta|2B)]^{-1} R(2w|\delta|2B) \exp(i2\pi\langle\delta, w\rangle), \tag{4}$$

$$R(2w|\delta|2B) = \sum_{(.)} \exp\{-i2\pi\langle\mu^s, [2w^s + \bar{B}^{ss}(\delta^s - \mu^s) + B^{sp}\delta^p]\rangle\}$$
$$\times\, \theta(2w^p + B^{pp}\delta^p + B^{ps}(\delta^s - 2\mu^s) \,|\, 2B^{pp}), \tag{5}$$

and the sum $(.)$ is over all $\mu^s \in D^s$: $(\delta^s - \mu^s) \in D^s$; $\delta = \delta^s + \delta^p$.

Equation (3) is an addition formula for T-fs which (for τ-functions) Jimbo *et al* (1981) were looking for. In the particular case when $C^s = 0$, (3) reduces to the addition theorem for θ-fs (Dubrovin 1981). In general, relation (3) gives convenient rules for the differentiation of T-fs and in our opinion its existence decides the utility of the famous Hirota bilinear operator methods (see appendix 2).

Denoting $L \equiv \ln T(z)$, the partial derivatives $(\partial_{z_i}(.) = (.)_{,i})$ are now

$$L_{,ij} = T^{-2}(z) \sum_{\alpha \in D^g} [\partial_{w_i}\partial_{w_j}\Omega_\alpha(0)] T^2(z+\alpha/2), \tag{6}$$

$$L_{,ijkl} + 2(L_{,ij}L_{,kl} + L_{,ik}L_{,jl} + L_{,il}L_{,jk})$$
$$= T^{-2}(z) \sum_{\alpha \in D^g} [\partial_{w_i} \ldots \partial_{w_l}\Omega_\alpha(0)] T^2(z+\alpha/2), \qquad \text{etc.} \tag{7}$$

2. Example of the application of the method

In the sequel we restrict ourselves to the KdV equation. Starting from the ansatz

$$u = W + L_{,xx}, \qquad W \in R - \text{constant}, \tag{8}$$

the KdV equation $u_{,t} = (u_{,xx} + 6u^2)_{,x}$ reads as $L_{,yt} = L_{,yyyy} + 6(L_{,yy})^2 + C$, with $y = x + 12Wt$ and with a new constant $C \in R$.

If $z = \varkappa y + \omega t + z_0$; $\varkappa$, ω, $z_0 \in C^g$, the last differential equation, if it holds for any z_0, by (6) and (7) leads to the system of algebraic equations which is in fact a system of dispersion equations for the KdV equation

$$\sum_{i,j=1}^{g} f_{ij}(\delta)\varkappa_i\omega_j = \sum_{i,j,k,l=1}^{g} [f_{ijkl}(\delta)\varkappa_i\varkappa_j\varkappa_k\varkappa_l] + 2Cf_0, \tag{9}$$

for any $\delta \in D^g$. The coefficients $f_{i,\ldots,l}$ are given by

$$f_{i,\ldots,l}(\delta) = \partial_{z_i}, \ldots, \partial_{z_l}\Bigg(\sum_{(.)} \exp i2\pi[\langle(\delta^s - 2\mu^s), z^s\rangle + \langle(\mu^s - \delta^s), \tilde{B}^{ss}\mu^s\rangle$$

$$-\langle\delta^p, B^{ps}\mu^s\rangle]\theta(2z^p + B^{ps}(\delta^s - 2\mu^s) + B^{pp}\delta^p | 2B^{pp})\Bigg)\Bigg|_{z=0}, \tag{10}$$

and the sum is over μ^s as in (5). Derivation is elementary and (9) has the same form as for periodic processes because of (3), but only the coefficients are different.

If (9) has a non-trivial solution, with respect to $\varkappa, \omega \in D^g$, the KdV equation has a solution (8), (with constant background W as this was discussed in the pure soliton case by Au and Fung (1982)).

For simplicity we assume $W = 0$. The dispersion equations (9) represent a set of 2^g equations with respect to $2g+1$ quantities $(\varkappa, \omega, C)$ and in the particular case of the pure quasi-periodic solutions $(C^s = 0)$ have been exhaustively discussed by Dubrovin (1981) for $g \leq 2$.

On the other hand, if $C^p = 0$, the class of solutions of (9) contains the multi-soliton solutions (regular and real) which indicates that one can look for solutions for arbitrary g.

In the case of $C^p = 0$ putting e.g. $\delta = 0$, $\delta = (0, \ldots, 0, 1, 0, \ldots, 0)$, and $\delta = (0, ., 0, 1, 0, .0, 1, 0, .0)$ with the units on the pth and qth place one finds $C = 0$, $\omega_p = (i2\pi)^2\varkappa_p^3$ and $\exp(i\pi B_{pq}) = (\varkappa_p + \varkappa_q)^2/(\varkappa_p - \varkappa_q)^{-2}$, respectively; i.e. the soliton relations in Hirota's (1980) version.

For present purposes the most important cases are the intermediate situations. We confine ourselves to $C^s = C^p = C^1$ i.e. to the soliton on the background of a periodic wave (plus constant background provided $W \neq 0$). In this simplest case, solutions have also been found by the technique of Bäcklund transformation (Wahlquist 1976) and by the inverse scattering method (Kuznetsov and Mikhailov 1974) but the T-f formalism presented here seems to be more effective in general. The T-f now contains only two terms and the solution is

$$u = W + \{\ln[\theta(z_2 | B_{22}) + \theta(z_2 + B_{12} | B_{22}) \exp i\pi(2z_1 + \mathrm{Re}\, B_{11})]\}_{,xx} \tag{11}$$

with $\varkappa_{1,2}$ and $\omega_{1,2}$ determined by the dispersion equations (9).

If $z_1, z_2, B_{ij} \in iR$, $i, j = 1, 2$ we have real solutions and it can be seen that the two terms in (11) describe the periodic wavetrain before and after interaction with the soliton; as a result the wavetrain is shifted in phase only (by the off-diagonal element of the B-matrix).

According to (2), the T-f formalism provides a generalisation of the Riemann θ-f so the knowledge when a (pure) quasi-periodic solution has as a limit the multisolution solution determined by the T-f formalism is important. If the limit of the quasi-periodic solution is defined in the sense of equation (1) and exists, one requires that $z^s \in iR$ (in the opposite case either singular or complex solutions appear). But for $g > 2$ the problem becomes of the same type as in the case of pure quasi-periodic solutions where conditions on the B-matrix can appear (Dubrovin 1981).

As a final remark, let us note that the simplicity of the T-f technique is closely related to the dispersion equation method since in the language of contour integrals (Matveev 1979, Ablowitz and Segur 1981), when the finite gap spectrum tends to a discrete one, some integrals become divergent creating trouble in the description of the limiting case.

Acknowledgments

The author acknowledges helpful discussions with M Jaworski.

Appendix 1. *T*-functions

For the Riemannian matrix $B \in C^{g\times g}$, which is symmetric and with positively defined imaginary part, the θ-function is defined as

$$\theta(z\,|\,B)=\sum_{n\in Z^g}\exp[\mathrm{i}\pi(2\langle n,z\rangle+\langle n,Bn\rangle)], \tag{A1.1}$$

(e.g. Igusa 1972, Dubrovin 1981, Zagrodzinski 1982, 1983). Hence according to (1)

$$\begin{aligned}T\begin{pmatrix} z^s-\mathrm{i}D^{ss}e^s/2\\ z^p\end{pmatrix}&=\sum_{n^s\in Z^s}\sum_{n^p\in Z^p}\exp\{\mathrm{i}\pi[2\langle n^s,(z^s-\mathrm{i}D^{ss}e^s/2)\rangle\\&\quad+2\langle n^p,z^p\rangle+\langle (n^s+n^p),B(n^s+n^p)\rangle]\}\\&=\sum_{n^s\in Z^s}\exp\{\mathrm{i}\pi[2\langle n^s,z^s\rangle+\langle n^s,\bar B^{ss}n^s\rangle]\}\exp[-\pi\langle n^s,D^{ss}(n^s-e^s)\rangle]\\&\quad\times\sum_{n^p\in Z^p}\exp\{\mathrm{i}\pi[2\langle n^p,(z^p+B^{ps}n^s)\rangle+\langle n^p,B^{pp}n^p\rangle]\}.\end{aligned} \tag{A1.2}$$

The last sum over Z^p is equal to $\theta(z^p+B^{ps}n^s\,|\,B^{pp})$. The second term in (A1.2) in the limit, becomes

$$\lim_{D^{ss}\to\infty}\exp\,[-\pi\langle n^s,D^{ss}(n^s-e^s)\rangle]=\begin{cases}1, & \text{if } n^s\in D^s\\ 0, & \text{if } n^s\notin D^s,\end{cases} \tag{A1.3}$$

and therefore the sum over Z^s reduces to the sum over D^s, i.e. $(n^s)_i=0$ or 1. Thus we obtain

$$T(z)=\sum_{n^s\in D^s}\exp(\mathrm{i}\pi[2\langle n^s,z^s\rangle+\langle n^s,\bar B^{ss}n^s\rangle])\,\theta(z^p+B^{ps}n^s|B^{pp}). \tag{A1.4}$$

Equations (3)–(5) follow from the addition theorem for the θ-functions

$$\theta(z+w\,|\,B)\theta(z-w\,|\,B)=2\sum_{\alpha\in D^g}\bar\Omega_\alpha(w)\theta^2(z+\alpha/2\,|\,B) \tag{A1.5}$$

with

$$\bar\Omega_\alpha(w)=2^{-(g+1)}\sum_{\alpha\in D^g}(-1)^{\langle\alpha,\delta\rangle}\theta(B\delta\,|\,2B)^{-1}\theta(2w+B\delta\,|\,2B)\exp(\mathrm{i}2\pi\langle\delta,w\rangle), \tag{A1.6}$$

provided that the limiting procedure as defined in (1) is performed.

For the particular case of $C^s\equiv 0$, (3) is equivalent once again to (A1.5). In the opposite situation when $C^p\equiv 0$, by (2) and (5) we have

$$T(z)=\sum_{\alpha\in D^s}\exp[\mathrm{i}\pi(2\langle\alpha,z^s\rangle+\langle\alpha,\bar B^{ss}\alpha\rangle)], \tag{A1.7}$$

$$R(2w|\delta^s|2B)=\sum_{\mu^s:(\delta^s-\mu^s)\in D^s}\exp\{-\mathrm{i}2\pi\langle\mu^s,[2w^s+\bar B^{ss}(\delta^s-\mu^s)]\rangle\}, \tag{A1.8}$$

respectively, under the condition Im Diag $\bar B^{ss}=0$.

In this way θ-functions represent a particular case of T-functions, but on the other hand T-functions can be treated as a limiting case of θ-functions but of a higher order. Note that (A1.7) coincides with the f-function of Hirota (1980, p 164).

Appendix 2. The Hirota bilinear operator and addition theorem for T-functions

The famous Hirota bilinear operator D is defined by

$$D_{z_{i_1}} \ldots D_{z_{i_q}} f \circ g := (\partial_{z_{i_1}} - \partial_{z'_{i_1}}) \ldots (\partial_{z_{i_q}} - \partial_{z'_{i_q}}) f(z) g(z')|_{z=z'}, \tag{A2.1}$$

where z_i is the ith component of $z \in C^g$, $1 \leq i_1, \ldots, i_q \leq q$ and the indices can be repeated (e.g. Hirota 1980, Ablowitz and Segur 1981, Rogers and Shadwick 1982).

From (A2.1) it is seen that

$$D_{z_{i_1}} \ldots D_{z_{i_q}} f \circ g = \partial_{w_{i_1}} \ldots \partial_{w_{i_q}} f(z+w) g(z-w)|_{w=0}. \tag{A2.2}$$

Moreover, if $f = g = T$, where T is given by (1) or (2) and the $D \ldots D$ operator is of even order, i.e. $q = 2s$, by the addition formula (3) we get also the representation

$$D_{z_{i_1}} \ldots D_{z_{i_{2s}}} T \circ T = \sum_{\alpha \in D^g} [\partial_{w_{i_1}} \ldots \partial_{w_{i_{2s}}} \Omega_\alpha(0)] T^2(z + \alpha/2), \tag{A2.3}$$

which shortly we will write as

$$D^{2s} T \circ T = \sum_\alpha (d^{2s} \Omega_\alpha) T^2(z + \alpha/2). \tag{A2.4}$$

Now observe that most of the soliton type differential equations in the Hirota formalism leads to the form (Hirota 1980)

$$\sum_{\beta \in A} \sum_{s=0}^{g} \sum_{\substack{p,q,r: \\ p+q+r=2s}} (a_{pqr\beta} D_x^p D_y^q D_t^r) F_\beta \circ F_\beta = 0 \tag{A2.5}$$

where $a_{pqr\beta}$ are constant and $z = \varkappa x + \nu y + \omega t + z_0 \in C^g$ and A represents a finite set.

For example, the KdV equation reduces to $D_x(D_t + D_x^3) F \circ F = 0$, the Kadomtzev–Petviashvili equation to $(D_y^2 + D_x^4 + D_x D_t) F \circ F = 0$, the Kotera-Sawada equation to $D_x(D_t + D_x^5) F \circ F = 0$, the sine-Gordon equation to $(C + D_t^2 - D_x^2) F_1 \circ F_1 = F_2^2$; $F_2 = F_1^*$, $C \in R$, etc. Observe that all these above differential operators are of the even order.

By identifying F_β with $T_\beta \equiv T(z + \beta/2)$ and hence treating A as a subset of D^g, we find that if $z \in C^g$, and after a change of the independent variables $x, y, t \to z_1, \ldots, z_g$, equation (A2.5) becomes

$$\sum_{\beta \in A} \sum_s G_s(D^{2s}) T_\beta \circ T_\beta = 0. \tag{A2.6}$$

Here $G_s(D^{2s})$ denotes a multilinear form of all possible D^{2s} operators of order $2s$ with coefficients which are polynomials in $\varkappa$'s, ν's and ω's.

According to (A2.3), (A2.6) can be simplified to

$$\sum_{\beta \in A} \sum_{\alpha \in D^g} \sum_s G_s(d^{2s} \Omega_\alpha) T^2[z + (\alpha + \beta)/2] = 0,$$

and next to

$$\sum_{\alpha \in D^g} \sum_{\beta \in A} \sum_s G_s(d^{2s} \Omega_{\varepsilon - \beta}) T^2(z + \varepsilon/2) = 0, \tag{A2.7}$$

which will be satisfied if the system of 2^g algebraic equations

$$\sum_{\beta\in A}\sum_{s} G_s(d^{2s}\Omega_{\varepsilon-\beta})=0 \qquad \text{for each } \varepsilon\in D^g, \tag{A2.8}$$

holds (the T_ε^2 functions form a set of linearly independent functions of z).

The expressions in (A2.8) are the polynomials in $\varkappa_i$, ν_i, and ω_i, $1\leqslant i\leqslant g$, and this means of course that (A2.8) is the system of dispersion equations considered above if the ansatz containing $\Sigma_{\beta\in A}\, c_\beta \ln T_\beta$ and/or its derivatives is substituted into the original partial differential equation.

Indeed,

$$\partial_{w_{i_1}}\ldots\partial_{w_{i_{2s}}} T(z+w)T(z-w)|_{w=0}=T^2M_{1,\ldots,2s}(L), \tag{A2.9}$$

where $L=\ln T$. The second and fourth order M operators were given by (6) and (7), respectively. The 6th and 8th order ones are the following

$$M_{1,\ldots,6}(L)=L_{1,\ldots,6}+2(L_{1,2}L_{3,\ldots,6}+\underset{15\times}{\ldots},)+4(L_{1,2}L_{3,4}L_{5,6}+\underset{15\times}{\ldots},), \tag{A2.10}$$

$$M_{1,\ldots,8}(L)=L_{1,\ldots,8}+2(L_{1,2}L_{3,\ldots,8}+\underset{28\times}{\ldots},)+2(L_{1,\ldots,4}L_{5,\ldots,8}+\underset{35\times}{\ldots},)$$
$$+8(L_{1,2}L_{3,4}L_{5,6}L_{7,8}+\underset{105\times}{\ldots},)+4(L_{1,2}L_{3,4}L_{5,\ldots,8}+\underset{210\times}{\ldots},). \tag{A2.11}$$

Below each term the number of combinations is given. These operators are useful when the solutions of higher order KdV equations are considered (Caudrey 1978).

A few remarks must be made. It is clear that (A2.8), although derived here in the spirit of the Hirota technique, coincides with the dispersion equations (e.g. (9) for KdV) and by (A2.4) is an implicit consequence of the addition theorem (3), whose importance with respect to θ-functions was raised also by Jimbo *et al* (1981) and Date *et al* (1982).

If (A2.8) has a solution, the NLPD equation will be satisfied by the relevant ansatz of the T-functions describing processes more general than pure solitons (including also the singular and rational ones).

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